

COCENTERS AND REPRESENTATIONS OF AFFINE 0-HECKE ALGEBRAS

XUHUA HE AND SIAN NIE

ABSTRACT. In this paper, we study the relation between the co-center $\tilde{\mathcal{H}}_0$ and the finite dimensional representations of an affine 0-Hecke algebra $\tilde{\mathcal{H}}_0$. As a consequence, we obtain a new criterion on the supersingular modules: a (virtual) module of $\tilde{\mathcal{H}}_0$ is supersingular if and only if its character vanishes on the non-supersingular part of $\tilde{\mathcal{H}}_0$.

INTRODUCTION

0.1. Extended affine Hecke algebras $\tilde{\mathcal{H}}_q$ are deformations of the group algebras of extended affine Weyl groups \tilde{W} (with the parameter function q). They play an important role in the study of representations of p -adic groups G .

For complex representations, Borel correspondence relates the representation of G with Iwahori fixed points to representations of $\tilde{\mathcal{H}}_q$, where q is a power of the prime number p .

For representations in characteristic p (the defining characteristic), Vignéras [17] relates the representations of G with representations of affine 0-Hecke algebras $\tilde{\mathcal{H}}_0$ and its generalization, pro- p Iwahori-Hecke algebras.

0.2. By the work of Kazhdan-Lusztig [11] and Reeder [15], the simple modules of $\tilde{\mathcal{H}}_q$ (for q nonzero and not a root of unity) are parameterized by the triple (s, u, ϕ) , where s is a semisimple element in the dual group G^\vee , u is a unipotent element in G^\vee with $sus^{-1} = u^q$ and ϕ is a local system of Springer type.

The classification of simple modules of $\tilde{\mathcal{H}}_0$, on the other hand, looks quite different. Abe [1] gave a classification of mod- p representations in terms of parabolic inductions of simple supersingular modules. Olivier [14] and Vignéras [20] classified all simple supersingular modules in terms of supersingular characters. The proof uses Bernstein presentation [18] and Satake-type isomorphism [19].

Key words and phrases. affine Coxeter groups, 0-Hecke algebras, Conjugacy classes.

0.3. In this paper, we study the cocenter $\overline{\tilde{\mathcal{H}}_0}$ of $\tilde{\mathcal{H}}_0$ and the trace map $Tr : \overline{\tilde{\mathcal{H}}_0} \rightarrow R(\tilde{\mathcal{H}}_0)_{\mathbf{k}}^*$ induced from the natural trace pairing between the cocenter $\overline{\tilde{\mathcal{H}}_0}$ and the Grothendieck group $R(\tilde{\mathcal{H}}_0)_{\mathbf{k}}$ of finite dimensional representations of $\tilde{\mathcal{H}}_0$ over an arbitrary algebraically closed field \mathbf{k} . We then use the trace map to give a basis of $R(\tilde{\mathcal{H}}_0)$. As a consequence, we give a new proof of the classification of simple supersingular modules of $\tilde{\mathcal{H}}_0$.

In the rest of the introduction, we explain our main results and compare them with results for $\tilde{\mathcal{H}}_q$. For simplicity, we only state the results for the case that \tilde{W} is an affine Weyl group. In the body of the paper, we tackle the general case.

0.4. The affine 0-Hecke algebra $\tilde{\mathcal{H}}_0$ has a standard \mathbb{Z} -basis $\{T_{\tilde{w}}; \tilde{w} \in \overline{\tilde{W}}\}$ subject to quadratic relations and braid relations. For the cocenter $\overline{\tilde{\mathcal{H}}_0}$, we have the following basis Theorem.

Theorem 0.1. *The set $\{T_{\Sigma}; \Sigma \in Cyc(\tilde{W}_{\min})\}$ forms a \mathbb{Z} -basis of $\overline{\tilde{\mathcal{H}}_0}$.*

Here \tilde{W}_{\min} is the set of elements in the affine Weyl group \tilde{W} that are of minimal lengths in their conjugacy classes and $Cyc(\tilde{W}_{\min})$ is the set of cyclic-shift classes in \tilde{W}_{\min} (defined in §2.1). The element T_{Σ} is the image of $T_{\tilde{w}}$ in $\overline{\tilde{\mathcal{H}}_0}$ for some, or equivalently, any $\tilde{w} \in \Sigma$.

This result is obtained using some nice properties of \tilde{W}_{\min} established in [9] and an idea in [7] for finite 0-Hecke algebras.

It is interesting to compare the cocenter of $\tilde{\mathcal{H}}_0$ with that of $\tilde{\mathcal{H}}_q$ for $q \neq 0$. For the latter one, a similar result is obtained in [9] (for equal parameter case) and [3] (for general case). For $\tilde{\mathcal{H}}_q$, the cocenter has a basis indexed by the set of “strongly conjugacy classes” of \tilde{W}_{\min} , which is in natural bijection with the set of conjugacy classes of \tilde{W} .

0.5. Now we move to the trace map $Tr : \overline{\tilde{\mathcal{H}}_0} \rightarrow R(\tilde{\mathcal{H}}_0)_{\mathbf{k}}^*$ and discuss its application on representations of $\tilde{\mathcal{H}}_0$.

Using parabolic induction and the basis Theorem for the cocenter, we can essentially reduce the study of the trace map to the study of the trace map for the 0-Hecke algebras of parahoric subgroups. Notice that the 0-Hecke algebra of a parahoric subgroup is a finite 0-Hecke algebra, whose simple modules have been classified in [12]. We have

Theorem 0.2. *The set $\{\pi_{J,\Gamma,\chi}; (J, \Gamma) \in \mathbb{N}/\sim, \chi \in \Omega_J(\Gamma)^{\vee}\}$ is a \mathbb{Z} -basis of $R(\tilde{\mathcal{H}}_0)_{\mathbf{k}}$.*

Here $\pi_{J,\Gamma,\chi}$ is, roughly speaking, an $\tilde{\mathcal{H}}_0$ -module induced from certain simple module of the parabolic subalgebra $\tilde{\mathcal{H}}_{J,0}^+$, which is indexed by the character χ and the parahoric subalgebra of $\tilde{\mathcal{H}}_{J,0}^+$ of type Γ . We refer to §4.2 for the precise definition.

0.6. By combining Theorem 0.2 with the character formula (Theorem 4.4), we obtain in Proposition 5.4 a new proof of the classification of simple supersingular modules. We also obtain the following criterion of supersingular modules.

Theorem 0.3. *An element $\pi \in R(\tilde{\mathcal{H}}_0)_{\mathbf{k}}$ is supersingular if and only if $\text{Tr}(h, \pi) = 0$ for all $h \in \overline{\tilde{\mathcal{H}}_0}^{nss}$.*

Here $\overline{\tilde{\mathcal{H}}_0}^{nss}$ is the non-supersingular part of the cocenter, defined as the subspace of $\tilde{\mathcal{H}}_0$ spanned by T_{Σ} and $'T_{\Sigma}$, where Σ is not contained in any proper parahoric group of \tilde{W} and ι is an involution of $\tilde{\mathcal{H}}_0$ defined in §1.3.

0.7. Again, it is interesting to compare the above results on $R(\tilde{\mathcal{H}}_0)_{\mathbf{k}}$ to the results on $R(\tilde{\mathcal{H}}_q)_{\mathbb{C}}$ for generic $q \neq 0$.

The trace pairing $\text{Tr} : \overline{\tilde{\mathcal{H}}_q} \rightarrow R(\tilde{\mathcal{H}}_q)_{\mathbb{C}}^*$ and the cocenter-representation duality for $\tilde{\mathcal{H}}_q$ are studied in [3]. Using the parabolic induction, we are reduced to study the trace pairing between the so-called rigid cocenter and the rigid modules. Here the rigid cocenter is the subspace of $\overline{\tilde{\mathcal{H}}_q}$ spanned by the images all proper parahoric subalgebras. The rigid modules are constructed using Lusztig's reduction theorem from affine Hecke algebras to graded affine Hecke algebras, and Springer representations for the finite Weyl group in the corresponding graded affine Hecke algebras. It is proved in [3, Theorem 1.1] that such pairing is perfect.

For $\tilde{\mathcal{H}}_0$, as we have seen above, the situation is different. What appears in this situation is not the representations of finite Weyl groups (or equivalently, the finite Hecke algebras with generic parameters), but that of the finite 0-Hecke algebra instead. This provides an interpretation for the difference between the representation theory of $\tilde{\mathcal{H}}_q$ for $q \neq 0$ and that of $\tilde{\mathcal{H}}_0$.

0.8. The paper is organized as follows.

In section 1, we recall the definition of affine 0-Hecke algebras, parabolic algebras, and trace maps. In section 2, we describe the cocenters of extended affine 0-Hecke algebras. In section 3, we introduce the standard pairs and use them to compute the characters of $\tilde{\mathcal{H}}_0$ -modules. In section 4, we construct some finite-dimensional modules and provide some character formulas. In section 5, we give a basis of the Grothendieck group of finite dimensional modules and study rigid and supersingular modules.

1. PRELIMINARY

1.1. Let $\mathfrak{R} = (X, R, Y, R^{\vee}, F_0)$ be a based root datum, where X and Y are free abelian groups of finite rank together with a perfect pairing

$\langle , \rangle : X \times Y \rightarrow \mathbb{Z}$, $R \subseteq X$ is the set of roots, $R^\vee \subseteq Y$ is the set of coroots and $F_0 \subseteq R$ is the set of simple roots. Let $\alpha \mapsto \alpha^\vee$ be the natural bijection from R to R^\vee such that $\langle \alpha, \alpha^\vee \rangle = 2$. For $\alpha \in R$, we denote by $s_\alpha : X \rightarrow X$ the corresponding reflections stabilizing R . Let $R^+ \subseteq R$ be the set of positive roots determined by F_0 . Let $X^+ = \{\lambda \in X; \langle \lambda, \alpha^\vee \rangle \geq 0, \forall \alpha \in R^+\}$. For any $v \in X_{\mathbb{Q}}$, we set $J_v = \{\alpha \in F_0; \langle v, \alpha^\vee \rangle = 0\}$. For any $J \subseteq F_0$, we set $X^+(J) = \{\lambda \in X^+; J_\lambda = J\}$.

1.2. Let W_0 be the (finite) Weyl group generated by the set of simple reflections $S_0 = \{s_\alpha; \alpha \in F_0\}$.

Let $W_{\text{aff}} = \mathbb{Z}R \rtimes W_0$ be the affine Weyl group and $S_{\text{aff}} \supset S_0$ be the set of simple reflections in W_{aff} . Then $(W_{\text{aff}}, S_{\text{aff}})$ is a Coxeter group. Let $\tilde{W} = X \rtimes W_0$ be the extended affine Weyl group. Then W_{aff} is a subgroup of \tilde{W} . For $\lambda \in X$, we denote by $t^\lambda \in \tilde{W}$ the corresponding translation element.

Let $V = X \otimes_{\mathbb{Z}} \mathbb{R}$. For $\alpha \in R$ and $k \in \mathbb{Z}$, set

$$H_{\alpha,k} = \{v \in V; \langle v, \alpha^\vee \rangle = k\}.$$

Let $\mathfrak{H} = \{H_{\alpha,k}; \alpha \in R, k \in \mathbb{Z}\}$. Connected components of $V - \cup_{H \in \mathfrak{H}} H$ are called alcoves. Let

$$C_0 = \{v \in V; 0 < \langle v, \alpha^\vee \rangle < 1, \forall \alpha \in R^+\}$$

be the fundamental alcove. We may regard W_{aff} and \tilde{W} as subgroups of affine transformations of V , where t^λ acts by translation $v \mapsto v + \lambda$ on V . The actions of W_{aff} and \tilde{W} on V preserve the set of alcoves.

For any $\tilde{w} \in \tilde{W}$, we denote by $\ell(\tilde{w})$ the number of hyperplanes in \mathfrak{H} separating C_0 from $\tilde{w}C_0$. Then $\tilde{W} = W_{\text{aff}} \rtimes \Omega$, where $\Omega = \{\tilde{w} \in \tilde{W}; \ell(\tilde{w}) = 0\}$ is the subgroup of \tilde{W} stabilizing fundamental alcove C_0 . The conjugation action of Ω on \tilde{W} preserves the set S_{aff} of simple reflections in W_{aff} .

For any $x \in W_{\text{aff}}$ and any $\tau \in \Omega$, we define

$$\text{supp}(x\tau) = \cup_{i \in \mathbb{N}} \tau^i(\text{supp}(x))\tau^{-i}.$$

Here $\text{supp}(x)$ is the set of simple reflections that appear in some (or equivalently, any) reduced expression of x .

1.3. The (generic) Hecke algebra $\tilde{\mathcal{H}}_q$ associated to the extended affine Weyl group \tilde{W} is an associative $\mathbb{Z}[q]$ -algebra with basis $\{T_{\tilde{w}}; \tilde{w} \in \tilde{W}\}$ subject to the following relations

$$\begin{aligned} T_{\tilde{x}} T_{\tilde{y}} &= T_{\tilde{x}\tilde{y}}, \quad \text{if } \ell(\tilde{x}) + \ell(\tilde{y}) = \ell(\tilde{x}\tilde{y}); \\ (T_s + 1)(T_s - q) &= 0, \quad \text{for } s \in S_{\text{aff}}. \end{aligned}$$

If we set $q = 0$, then the second relation becomes $T_s^2 = -T_s$ and the \mathbb{Z} -algebra we obtain is called the (affine) 0-Hecke algebra associated to \tilde{W} . We denote it by $\tilde{\mathcal{H}}_0$.

By [17, Corollary 2], the map $T_{\tilde{w}} \mapsto {}^t T_{\tilde{w}} := (-q)^{\ell(\tilde{w})} T_{\tilde{w}^{-1}}$ gives an involution ι of $\tilde{\mathcal{H}}_q$. We still denote by ι the induced involution of $\tilde{\mathcal{H}}_0$.

1.4. Let $[\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]$ be the commutator of $\tilde{\mathcal{H}}_0$, the \mathbb{Z} -submodule spanned by $[T_{\tilde{x}}, T_{\tilde{y}}] := T_{\tilde{x}}T_{\tilde{y}} - T_{\tilde{y}}T_{\tilde{x}}$ for $\tilde{x}, \tilde{y} \in \tilde{W}$. Let $\overline{\tilde{\mathcal{H}}_0} = \tilde{\mathcal{H}}_0 / [\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]$ be the cocenter of $\tilde{\mathcal{H}}_0$. Denote by $R(\tilde{\mathcal{H}}_0)_{\mathbf{k}}$ the Grothendieck group of finite dimensional representations of $\tilde{\mathcal{H}}_0$ over an arbitrary algebraically closed field \mathbf{k} . Consider the trace map

$$Tr : \overline{\tilde{\mathcal{H}}_0} \rightarrow R(\tilde{\mathcal{H}}_0)_{\mathbf{k}}^*, \quad h \mapsto (V \mapsto Tr(h, V)).$$

Similar map for generic $q \in \mathbb{C}^\times$ and $\mathbf{k} = \mathbb{C}$ is studied in the joint work of Ciubotaru and the first-named author [3], in which case the trace map is injective and there is a “perfect pairing” between the rigid-cocenter and rigid-representations of $\tilde{\mathcal{H}}_q$.

For $q = 0$, the situation is different. The map is not injective. However, there is still a nice pairing between cocenter and representations.

1.5. Now we introduce parabolic subalgebras.

For any $J \subseteq F_0$, we denote by R_J the set of roots spanned by J and set $R_J^\vee = \{a^\vee; \alpha \in R_J\}$. Let $\mathfrak{R}_J = (X, R_J, Y, R_J^\vee, J)$ be the based root datum corresponding to J . Let $W_J \subseteq W_0$ and $\tilde{W}_J = X \rtimes W_J$ be the Weyl group and the extended affine Weyl group of \mathfrak{R}_J respectively. We say $\tilde{w} \in \tilde{W}_J$ is J -positive if $\tilde{w} \in t^\lambda W_J$ for some $\lambda \in X$ such that $\langle \lambda, \alpha \rangle \geq 0$ for $\alpha \in R^+ \setminus R_J$. Denote by \tilde{W}_J^+ the set of J -positive elements, which is a submonoid of \tilde{W}_J , see [2, Section 6] and [16, II.4].

We set $\mathfrak{H}_J = \{H_{\alpha, k} \in \mathfrak{H}; \alpha \in R_J, k \in \mathbb{Z}\}$ and $C_J = \{v \in V; 0 < \langle v, \alpha^\vee \rangle < 1, \alpha \in R_J^+\}$. For any $\tilde{w} \in \tilde{W}_J$, we denote by $\ell_J(\tilde{w})$ the number of hyperplanes in \mathfrak{H}_J separating C_J from $\tilde{w}C_J$.

Let $\tilde{\mathcal{H}}_{J,0}$ be the affine 0-Hecke algebra associated to \mathfrak{R}_J with standard basis $T_{\tilde{w}}^J$ for $\tilde{w} \in \tilde{W}_J$. Let $\tilde{\mathcal{H}}_{J,0}^+$ be the subalgebra of $\tilde{\mathcal{H}}_{J,0}$ spanned by $T_{\tilde{w}}^J$ for $\tilde{w} \in \tilde{W}_J^+$. We have a natural embedding

$$\tilde{\mathcal{H}}_{J,0}^+ \hookrightarrow \tilde{\mathcal{H}}_0, \quad T_{\tilde{w}}^J \mapsto T_{\tilde{w}}.$$

Notice that this embedding does not extend to an algebra homomorphism $\tilde{\mathcal{H}}_{J,0} \rightarrow \tilde{\mathcal{H}}_0$ since $T_{t^\lambda}^J$ for $\lambda \in X^+(J)$ is invertible in $\tilde{\mathcal{H}}_{J,0}$, but T_{t^λ} is not invertible in $\tilde{\mathcal{H}}_0$ unless $J = F_0$.

Let $(W_J)_{aff} = \mathbb{Z}R_J \rtimes W_J$ and $J_{aff} \supseteq J$ the set of simple reflections of $(W_J)_{aff}$. Then $\tilde{W}_J = (W_J)_{aff} \rtimes \Omega_J$, where $\Omega_J = \{\tilde{w} \in \tilde{W}_J; \ell_J(\tilde{w}) = 0\}$. We denote by $\mathcal{H}_{J,0}$ the 0-Hecke algebra associated to $(W_J)_{aff}$.

We denote by \tilde{W}^J (resp. ${}^J \tilde{W}$) the set of minimal coset representatives in \tilde{W}/W_J (resp. $W_J \backslash \tilde{W}$). For $J, K \subseteq F_0$, we simply write $\tilde{W}^J \cap {}^K \tilde{W}$ as ${}^K \tilde{W}^J$. We define ${}^J W_0, W_0^J$ and ${}^J W_0^K$ in a similar way.

2. COCENTER OF $\tilde{\mathcal{H}}_0$

2.1. For $\tilde{w}, \tilde{w}' \in \tilde{W}$ and $s \in S_{\text{aff}}$, we write $\tilde{w} \xrightarrow{s} \tilde{w}'$ if $\tilde{w}' = s\tilde{w}s$ and $\ell(\tilde{w}') \leq \ell(\tilde{w})$. We write $\tilde{w} \rightarrow \tilde{w}'$ if there exists a sequence $\tilde{w} = \tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_n = \tilde{w}'$ of elements in \tilde{W} such that for any k , $\tilde{w}_{k-1} \xrightarrow{s_k} \tilde{w}_k$ for some $s_k \in S_{\text{aff}}$. We write $\tilde{w} \approx \tilde{w}'$ if there exists $\tau \in \Omega$ such that $\tilde{w} \rightarrow \tau\tilde{w}'\tau^{-1}$ and $\tau\tilde{w}'\tau^{-1} \rightarrow \tilde{w}$ and we say that \tilde{w} and \tilde{w}' are in the same cyclic-shift class.

Note that \approx is an equivalence relation. Let $cl(\tilde{W})$ be the set of conjugacy classes of \tilde{W} . For any $\mathcal{O} \in cl(W)$, let \mathcal{O}_{\min} be the set of minimal length elements in \mathcal{O} . Since \approx is compatible with the length function, \mathcal{O}_{\min} is a union of cyclic-shift classes.

Let $\tilde{W}_{\min} = \sqcup_{\mathcal{O} \in cl(W)} \mathcal{O}_{\min}$ and $\text{Cyc}(\tilde{W}_{\min})$ the set of cyclic-shift classes in \tilde{W}_{\min} .

2.2. Now we introduce a partial order on $\text{Cyc}(\tilde{W}_{\min})$.

Let $w \in \tilde{W}$ and $\Sigma \in \text{Cyc}(\tilde{W}_{\min})$. We write $\Sigma \preceq \tilde{w}$ if there exists $\tilde{w}' \in \Sigma$ such that $\tilde{w}' \leq \tilde{w}$.

For $\Sigma, \Sigma' \in \text{Cyc}(\tilde{W}_{\min})$, we write $\Sigma' \preceq \Sigma$ if $\Sigma' \preceq \tilde{w}$ for some $\tilde{w} \in \Sigma$. By [4, Corollary 4.6], $\Sigma' \preceq \Sigma$ if and only if $\Sigma' \preceq \tilde{w}$ for any $\tilde{w} \in \Sigma$. In particular, \preceq is transitive, which defines a partial order on $\text{Cyc}(\tilde{W}_{\min})$.

We have the following result.

Proposition 2.1. *Let $\tilde{w} \in \tilde{W}$. Then*

(1) *The set $\{\Sigma \in \text{Cyc}(\tilde{W}_{\min}); \Sigma \preceq \tilde{w}\}$ contains a unique maximal element $\Sigma_{\tilde{w}}$.*

(2) *Let $s \in S_{\text{aff}}$ such that $\tilde{w} \rightarrow s\tilde{w}s$. Then*

$$\Sigma_{\tilde{w}} = \begin{cases} \Sigma_{s\tilde{w}s}, & \text{if } \ell(s\tilde{w}s) = \ell(\tilde{w}); \\ \Sigma_{s\tilde{w}}, & \text{if } \ell(s\tilde{w}s) < \ell(\tilde{w}). \end{cases}$$

A similar statement is proved in [7, Proposition 6.2 (1)] for finite Weyl groups. The same proof also works for extended affine Weyl groups.

We also have the following result, which follows directly from the definition of $\Sigma_{\tilde{w}}$.

Lemma 2.2. *Let $\tilde{w} \in \tilde{W}$ and $\tau \in \Omega$. Then $\Sigma_{\tilde{w}} = \Sigma_{\tau\tilde{w}\tau^{-1}}$.*

2.3. By definition, if $\tilde{w} \approx \tilde{w}'$, then the images of $T_{\tilde{w}}$ and $T_{\tilde{w}'}$ in $\overline{\tilde{\mathcal{H}}_0}$ are the same. In particular, for any $\Sigma \in \text{Cyc}(\tilde{W}_{\min})$, we denote by T_{Σ} the image of $T_{\tilde{w}}$ in $\overline{\tilde{\mathcal{H}}_0}$ for any $\tilde{w} \in \Sigma$. We also denote by $\ell(\Sigma)$ the length of any element in Σ .

Similar to the proof of [7, Proposition 6.2 (2)], we have that

Proposition 2.3. *Let $\tilde{w} \in \tilde{W}$. Then the image of $T_{\tilde{w}}$ in $\overline{\tilde{\mathcal{H}}_0}$ equals $(-1)^{\ell(w)-\ell(\Sigma_{\tilde{w}})} T_{\Sigma_{\tilde{w}}}$.*

We also need the following observation on the commutator of $\tilde{\mathcal{H}}_0$.

Lemma 2.4. *The \mathbb{Z} -module $[\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]$ is spanned by $[T_{\tilde{w}}, T_x]$ for $\tilde{w} \in \tilde{W}$ and $x \in S_{aff} \cup \Omega$.*

Proof. Let $[\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]'$ be the submodule of $[\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]$ spanned by $[T_{\tilde{w}}, T_x]$ for $x \in S_{aff} \cup \Omega$. It suffices to show that $[T_{\tilde{w}}, T_{\tilde{w}'}] \in [\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]'$ for any $\tilde{w}, \tilde{w}' \in \tilde{W}$.

We argue by induction on $\ell(\tilde{w}')$. If $\ell(\tilde{w}') = 0$, then it follows by definition. Let $k \geq 1$. Suppose that $[T_{\tilde{w}}, T_{\tilde{u}}] \in [\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]'$ for any \tilde{u} with $\ell(\tilde{u}) < k$. Let $s \in S_{aff}$ with $s\tilde{w}' < \tilde{w}'$. Then

$$[T_{\tilde{w}}, T_{\tilde{w}'}] = [T_{\tilde{w}}T_s, T_{s\tilde{w}'}] + [T_{s\tilde{w}'}T_{\tilde{w}}, T_s].$$

By inductive hypothesis, $[T_{\tilde{w}}, T_{\tilde{w}'}] \in [\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]'$. \square

2.4. Now we prove Theorem 0.1.

Let M be the free \mathbb{Z} -module with basis $\{[\Sigma]; \Sigma \in Cyc(\tilde{W}_{\min})\}$. Define a \mathbb{Z} -linear map

$$\psi : \tilde{\mathcal{H}}_0 \rightarrow M, \quad T_{\tilde{w}} \mapsto (-1)^{\ell(w) - \ell(\Sigma_{\tilde{w}})} [\Sigma_{\tilde{w}}].$$

Let $\tilde{w} \in \tilde{W}$ and $s \in S_{aff}$. We show that

(a) $[T_{\tilde{w}}, T_s] \in \ker \psi$.

If $\tilde{w} < \tilde{w}s$ and $\tilde{w} < s\tilde{w}$, then $\ell(\tilde{w}\sigma) = \ell(s\tilde{w})$ and $[T_{\tilde{w}}, T_s] = T_{\tilde{w}s} - T_{s\tilde{w}}$.

By Proposition 2.1 (2), $\Sigma_{\tilde{w}s} = \Sigma_{s\tilde{w}}$ and $\psi([T_{\tilde{w}}, T_s]) = (-1)^{\ell(\tilde{w})+1-\ell(\Sigma_{\tilde{w}s})} (\Sigma_{\tilde{w}s} - \Sigma_{s\tilde{w}}) = 0$.

If $\tilde{w}s, s\tilde{w} < \tilde{w}$, then $[T_{\tilde{w}}, T_s] = 0 \in \ker \psi$.

If $s\tilde{w} < \tilde{w} < \tilde{w}s$, then $[T_{\tilde{w}}, T_s] = T_{\tilde{w}s} + T_{\tilde{w}}$. By Proposition 2.1(2), $\Sigma_{\tilde{w}s} = \Sigma_{\tilde{w}}$ and $\psi([T_{\tilde{w}}, T_s]) = (-1)^{\ell(\tilde{w})+1-\ell(\Sigma_{\tilde{w}s})} \Sigma_{\tilde{w}s} + (-1)^{\ell(\tilde{w})-\ell(\Sigma_{\tilde{w}})} \Sigma_{\tilde{w}} = 0$.

If $\tilde{w}s < \tilde{w} < s\tilde{w}$, then $[T_{\tilde{w}}, T_s] = -T_{\tilde{w}} - T_{s\tilde{w}}$ and by Proposition 2.1(2), $\Sigma_{\tilde{w}} = \Sigma_{s\tilde{w}}$ and $\psi([T_{\tilde{w}}, T_s]) = (-1)^{\ell(\tilde{w})+1-\ell(\Sigma_{\tilde{w}})} \Sigma_{\tilde{w}} + (-1)^{\ell(\tilde{w})+2-\ell(\Sigma_{\tilde{w}})} \Sigma_{s\tilde{w}} = 0$.

Thus (a) is proved.

By Lemma 2.2, $[T_{\tilde{w}}, T_\tau] \in \ker \psi$ for any $\tilde{w} \in \tilde{W}$ and $\tau \in \Omega$. Thanks to Lemma 2.4, $[\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0] \subseteq \ker \psi$ and we have an induced map $\overline{\tilde{\mathcal{H}}_0} \rightarrow M$, which we still denote by ψ .

On the other hand, we have a well-defined \mathbb{Z} -linear map $\phi : M \rightarrow \overline{\tilde{\mathcal{H}}_0}$ which sends $[\Sigma]$ to T_Σ . It is easy to see that $\psi \circ \phi$ is the identity map. In particular, ϕ is injective. By Proposition 2.3, ϕ is also surjective. Thus ϕ is an isomorphism.

3. STANDARD PAIRS

3.1. Let $n_0 = \#W_0$. For any $\tilde{w} \in \tilde{W}$, $\tilde{w}^{n_0} = t^\lambda$ for some $\lambda \in X$. We set $\nu_{\tilde{w}} = \lambda/n_0 \in X_{\mathbb{Q}}$ and $\bar{\nu}_{\tilde{w}} \in X_{\mathbb{Q}}^+$ the unique dominant element in

the W_0 -orbit of $\nu_{\tilde{w}}$. It is easy to see that the map $\tilde{W} \rightarrow V, \tilde{w} \mapsto \bar{\nu}_{\tilde{w}}$ is constant on each conjugacy class of \tilde{W} .

We say that an element $\tilde{w} \in \tilde{W}$ is straight if $\ell(\tilde{w}^n) = n\ell(\tilde{w})$ for any $n \in \mathbb{N}$. By [6, Lemma 1.1], \tilde{w} is straight if and only if $\ell(\tilde{w}) = \langle \bar{\nu}_{\tilde{w}}, 2\rho^\vee \rangle$, where ρ is the half sum of positive coroots. A conjugacy class that contains a straight element is called a straight conjugacy class.

It is proved in [9, Proposition 2.8] that for each cyclic-shift class in \tilde{W}_{\min} , we have some representatives as follows.

Proposition 3.1. *For any $\tilde{w} \in \tilde{W}_{\min}$, there exists a subset $K \subseteq S_{\text{aff}}$ with W_K finite, a straight element $y \in {}^K\tilde{W}^K$ with $yKy^{-1} = K$ and an element $w \in W_K$ such that $\tilde{w} \approx wy$. Here $W_K \subseteq W_{\text{aff}}$ denotes the subgroup generated by reflections of K .*

3.2. In the situation of Proposition 3.1, we call wy a standard representative of the cyclic-shift class of \tilde{w} . By [6, Proposition 2.2], $\bar{\nu}_{\tilde{w}} = \bar{\nu}_{wy} = \bar{\nu}_y$. The expression of standard representative relates each conjugacy class of \tilde{W} with a straight conjugacy class. It plays an important role in the study of combinatorial properties of conjugacy classes of affine Weyl groups [9], σ -conjugacy classes of p -adic groups [5] and representations of affine Hecke algebras with nonzero parameters [3].

However, for a given cyclic-shift class in \tilde{W}_{\min} , the standard representatives are in general, not unique. This leads to some difficulty in understanding the cyclic-shift classes in \tilde{W}_{\min} and their relations to the representations of \mathcal{H}_0 .

3.3. To overcome the difficulty, we introduce the standard pairs as follows.

Let wy be a standard representative as above. Then the conjugation by y sends simple reflections in $\text{supp}(w)$ to simple reflections. Set $K = \bigcup_{i \in \mathbb{N}} y^i \text{supp}(w) y^{-i}$. It is easy to see that K is the smallest subset of S_{aff} that $yKy^{-1} = K$ and $y \in {}^K\tilde{W}^K$.

Set $J = J_{\bar{\nu}_y}$. Let $z \in {}^J W_0$ with $z(\nu_y) = \bar{\nu}_y$. Set $x = zyz^{-1}$ and $\Gamma = zKz^{-1}$. Then $\Gamma \subseteq J_{\text{aff}}$ by noticing that $zC_0 \subseteq C_J$ (see §1.2 and §1.5).

It is easy to see that $\nu_x = \bar{\nu}_y \in X_{\mathbb{Q}}^+$, $\#W_\Gamma < +\infty$ and $x\Gamma x^{-1} = \Gamma$. We say that (x, Γ) is a standard pair associated to (the cyclic-shift class of) \tilde{w} .

Remark. There might be more than one standard pairs associated to a given cyclic-shift class. However, we will see by Theorem 4.4 that all these standard pairs are equivalent. Here we say two standard pairs (x, Γ) and (x', Γ') are equivalent if $x = x'$ and $\Gamma' = \omega\Gamma\omega^{-1}$ for some $\omega \in \Omega_{J_{\nu_x}}$.

Lemma 3.2. *Let wy be a standard representative and $K = \cup_{i \in \mathbb{N}} y^i \text{supp}(w) y^{-i}$. Then for $n \gg 0$,*

$$T_{wy}^n = (-1)^{n\ell(w)-\ell(w_K)} T_{w_K y^n},$$

where w_K is the maximal element in W_K .

Remark. Note that $w_K y^n \neq (wy)^n$. However, we may regard $w_K y^n$ as the n -th Demazure product of wy .

Proof. Let δ be the automorphism on W_K induced by the conjugation action of y . Let m be the order of the element $w\delta$ in $W_K \rtimes \langle \delta \rangle$. By [8, Corollary 5.5], $T_w \delta(T_w) \cdots \delta^{m-1}(T_w) = T_{w_K} T_{w_1} \cdots T_{w_l}$ for some $w_1, \dots, w_l \in W_K$ with $\ell(w_1) + \cdots + \ell(w_l) = m\ell(w) - \ell(w_K)$. Thus for any $n \geq m$,

$$\begin{aligned} T_w \delta(T_w) \cdots \delta^{n-1}(T_w) &= T_{w_K} T_{w_1} \cdots T_{w_l} \delta^m(T_w) \cdots \delta^{n-1}(T_w) \\ &= (-1)^{\ell(w_1)+\ell(w_l)+(n-m)\ell(w)} T_{w_K} \\ &= (-1)^{n\ell(w)-\ell(w_K)} T_{w_K}. \end{aligned}$$

Here the second equality follows from the definition of 0-Hecke algebras (as $T_{w_K} T_x = (-1)^{\ell(x)} T_{w_K}$ for any $x \in W_K$). Since y is a straight element, $T_{wy}^n = T_w \delta(T_w) \cdots \delta^{n-1}(T_w) T_y^n = (-1)^{n\ell(w)-\ell(w_K)} T_{w_K} T_y^n = (-1)^{n\ell(w)-\ell(w_K)} T_{w_K y^n}$. \square

The following result is a variation of the length formula in [10].

Lemma 3.3. *For $w \in W_0$ and $\alpha \in R$, set*

$$\delta_w(\alpha) = \begin{cases} 0, & \text{if } w\alpha \in R^+; \\ 1, & \text{if } w\alpha \in R^-. \end{cases}$$

Then for any $x, y \in W_0$ and $\mu \in X$, we have that

$$\ell(xt^\mu y) = \sum_{\alpha \in R^+} |\langle \mu, \alpha^\vee \rangle + \delta_x(\alpha) - \delta_{y^{-1}}(\alpha)|.$$

Proposition 3.4. *Let (x, Γ) be a standard pair. Then*

- (1) *for $n \gg 0$ and $u \in J_{\nu_x} W_0$, $\ell(u^{-1} w_\Gamma x^n u) = \ell(w_\Gamma x^n)$.*
- (2) *for $n \gg 0$, $\ell(w_\Gamma x^{n+n_0}) = \ell(w_\Gamma x^n) + \ell(x^{n_0})$, where $n_0 = \#W_0$.*

Here $w_\Gamma \in W_\Gamma \subseteq (W_J)_{aff}$ is the unique element with maximal length with respect to ℓ_J .

Proof. Set $J = J_{\nu_x}$. We have $w_\Gamma x^n = t^\lambda w$ for some $\lambda \in X$ and $w \in W_J$. Since $\langle \nu_x, \alpha^\vee \rangle > 0$ for any $\alpha \in F_0 \setminus J$, we have $\langle \lambda, \alpha^\vee \rangle > 0$ for any $\alpha \in R^+ \setminus R_J^+$ as $n \gg 0$.

Notice that for $\alpha \in R_J$, $\delta_{u^{-1}(\alpha)} = \delta_\alpha$. Now

$$\begin{aligned}
\ell(u^{-1}w_\Gamma x^n u) &= \sum_{\alpha \in R^+} |\langle \lambda, \alpha^\vee \rangle + \delta_{u^{-1}}(\alpha) - \delta_{u^{-1}w^{-1}}(\alpha)| \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee \rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \setminus R_J^+} |\langle \lambda, \alpha^\vee \rangle + \delta_{u^{-1}}(\alpha) - \delta_{u^{-1}w^{-1}}(\alpha)| \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee \rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \setminus R_J^+} (\langle \lambda, \alpha^\vee \rangle + \delta_{u^{-1}}(\alpha) - \delta_{u^{-1}w^{-1}}(\alpha)) \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee \rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \setminus R_J^+} \langle \lambda, \alpha^\vee \rangle + \#\{\alpha \in R^+ \setminus R_J^+, u^{-1}(\alpha) \in R^-\} \\
&\quad - \#\{\alpha \in R^+ \setminus R_J^+, u^{-1}w^{-1}(\alpha) \in R^-\} \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee \rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \setminus R_J^+} \langle \lambda, \alpha^\vee \rangle + \ell(u) - \ell(u) \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee \rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \setminus R_J^+} \langle \lambda, \alpha^\vee \rangle.
\end{aligned}$$

This proves part (1).

For part (2),

$$\begin{aligned}
\ell(w_\Gamma x^{n+n_0}) &= \sum_{\alpha \in R^+} |\langle \lambda + n_0 \nu_x, \alpha^\vee \rangle - \delta_{w^{-1}}(\alpha)| \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda + n_0 \nu_x, \alpha^\vee \rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \setminus R_J^+} \langle \lambda + n_0 \nu_x, \alpha^\vee \rangle \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee \rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \setminus R_J^+} \langle \lambda, \alpha^\vee \rangle + \sum_{\alpha \in R^+ \setminus R_J^+} \langle n_0 \nu_x, \alpha^\vee \rangle \\
&= \ell(w_\Gamma x^n) + \ell(x^{n_0}).
\end{aligned}$$

□

As a consequence, we have

Corollary 3.5. *Let (x, Γ) be a standard pair associated to the standard representative wy and $K = \cup_{i \in \mathbb{N}} y^i \text{supp}(w)y^{-i}$. Then for $n \gg 0$, $w_\Gamma x^n \tilde{\approx} w_K y^n$.*

Proof. Suppose that $z = s_1 \cdots s_k$ for $s_1, \dots, s_k \in S_0$. Set $z_i = s_1 \cdots s_i$ for $1 \leq i \leq k$. Then $z_i \in {}^{J_{\nu_x}} W_0$. By Proposition 3.4 (1), $\ell(z_i^{-1} w_\Gamma x^n z_i) = \ell(z_{i+1}^{-1} w_\Gamma x^n z_{i+1})$ for $0 \leq i \leq k-1$. Hence $z_i^{-1} w_\Gamma x^n z_i \tilde{\approx} z_{i+1}^{-1} w_\Gamma x^n z_{i+1}$ for $0 \leq i \leq k-1$. Therefore $w_\Gamma x^n \tilde{\approx} z^{-1} w_\Gamma x^n z = w_K y^n$. □

Now we show that the character of $T_{\tilde{w}}$ for $\tilde{w} \in \tilde{W}_{\min}$ is determined by standard pairs associated to \tilde{w} .

Proposition 3.6. *Let $\tilde{w} \in \tilde{W}_{\min}$ and (x, Γ) be a standard pair associated to \tilde{w} . Then for $n \gg 0$,*

$$\text{Tr}(T_{\tilde{w}}^n, \pi) = (-1)^{n\ell(\tilde{w}) - n\ell(x) - \ell(w_\Gamma)} \text{Tr}(T_{w_\Gamma x^n}, \pi)$$

for any $\pi \in R(\tilde{\mathcal{H}}_0)_k$.

Proof. Let wy be a standard representative of \tilde{w} . Then $wy \approx \tilde{w}$ and by definition, $T_{wy}^m - T_{\tilde{w}}^m \in [\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]$ for any $m \geq 0$. By Lemma 3.2 and Lemma 3.5, for $n \gg 0$,

$$T_{wy}^n = (-1)^{n\ell(w) - \ell(w_K)} T_{w_K y^n} \in (-1)^{n\ell(w) - \ell(w_K)} T_{w_\Gamma x^n} + [\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0].$$

Notice that $\ell(w) = \ell(\tilde{w}) - \ell(y) \equiv \ell(\tilde{w}) - \ell(x) \pmod{2}$ and $\ell(w_K) \equiv \ell(w_\Gamma) \pmod{2}$. Thus $T_{\tilde{w}}^n \in (-1)^{n\ell(\tilde{w}) - n\ell(x) - \ell(w_\Gamma)} T_{w_\Gamma x^n} + [\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_0]$. \square

Corollary 3.7. *Let $\tilde{w}, \tilde{w}' \in \tilde{W}_{\min}$ such that there is a common standard pair associated to them. Then $\text{Tr}(T_{\tilde{w}}, \pi) = \text{Tr}(T_{\tilde{w}'}, \pi)$ for any $\pi \in R(\tilde{\mathcal{H}}_0)_k$.*

Remark. Notice that elements in different conjugacy classes may have the same standard pair.

Proof. By Proposition 3.6, $\text{Tr}(T_{\tilde{w}}^n, \pi) = \text{Tr}(T_{\tilde{w}'}^n, \pi)$ for $n \gg 0$. Thus the action of $T_{\tilde{w}}$ and $T_{\tilde{w}'}$ on π have the same generalized eigenvalues with the same multiplications. Therefore $\text{Tr}(T_{\tilde{w}}, \pi) = \text{Tr}(T_{\tilde{w}'}, \pi)$. \square

4. CHARACTER FORMULAS

4.1. Let $M \in R(\tilde{\mathcal{H}}_0)_k$. For any $J \subseteq F_0$, we set $M_J = \cap_{\lambda \in X^+(J)} T_{t^\lambda} M$. Since M is a linear combination of finite dimensional vector spaces, there exists $\mu \in X^+(J)$ such that $M_J = T_{t^\mu} M$. Moreover, since the action of T_{t^λ} on M_J is invertible for any $\lambda \in X^+(J)$, we may regard M_J as an $\tilde{\mathcal{H}}_{J,0}$ -module. For $\Gamma \subseteq J_{\text{aff}}$, let $\Omega_J(\Gamma) = \{\tau \in \Omega_J; \tau \Gamma \tau^{-1} = \Gamma\}$ and $M_{J,\Gamma} = T_{w_\Gamma}^J M_J$. Then $M_{J,\Gamma}$ is an $\Omega_J(\Gamma)$ -module.

Lemma 4.1. *Let $\tilde{w} \in \tilde{W}_{\min}$ with an associated standard pair (x, Γ) and $M \in R(\tilde{\mathcal{H}}_0)_k$. Then for $n \gg 0$,*

$$\text{Tr}(T_{\tilde{w}}^n, M) = (-1)^{n\ell(\tilde{w}) - n\ell(x)} \text{Tr}((T_x^{J_{\nu_x}})^n, M_{J_{\nu_x}, \Gamma}).$$

$$\text{In particular, } \text{Tr}(T_{\tilde{w}}, M) = (-1)^{\ell(\tilde{w}) - \ell(x)} \text{Tr}(T_x^{J_{\nu_x}}, M_{J_{\nu_x}, \Gamma}).$$

Proof. Set $J = J_{\nu_x}$. Let $\mu \in X^+(J)$ with $M_J = T_{t^\mu} M$. Notice that $n_0 \nu_x \in X^+(J)$, where $n_0 = \#W_0$. There exists $m \in \mathbb{N}$ such that $mn_0 \nu_x - \mu \in X^+(J)$. By Proposition 3.4 (2), for $n \gg 0$, $\ell(w_\Gamma x^{n+mn_0}) = \ell(w_\Gamma x^n) + \ell(t^{mn_0 \nu_x}) = \ell(w_\Gamma x^n) + \ell(t^{mn_0 \nu_x - \mu}) + \ell(t^\mu)$ and

$$T_{w_\Gamma x^{n+mn_0}} = T_{w_\Gamma x^n} T_{t^{mn_0 \nu_x - \mu}} T_{t^\mu}.$$

Moreover, for $n \gg 0$, $w_\Gamma x^{n+mn_0} \in \tilde{W}_J^+$ and $T_{w_\Gamma x^{n+mn_0}} = T_{w_\Gamma x^{n+mn_0}}^J$. Since $0 \rightarrow \ker(T_{t^\mu} : M \rightarrow M) \rightarrow M \rightarrow M_J \rightarrow 0$, we have

$$\begin{aligned} Tr(T_{w_\Gamma x^{n+mn_0}}, M) &= Tr(T_{w_\Gamma x^n} T_{t^{mn_0\nu_x-\mu}} T_{t^\mu}, M_J) = Tr(T_{w_\Gamma x^{n+mn_0}}, M_J) \\ &= Tr(T_{w_\Gamma x^{n+mn_0}}^J, M_J). \end{aligned}$$

Notice that $T_{w_\Gamma x^{n+mn_0}}^J = T_{w_\Gamma}^J (T_x^J)^{n+mn_0} = (T_x^J)^{n+mn_0} T_{w_\Gamma}^J$. Since $0 \rightarrow \ker(T_{w_\Gamma} : M_J \rightarrow M_J) \rightarrow M_J \rightarrow M_{J,\Gamma} \rightarrow 0$, we have $Tr(T_{w_\Gamma x^{n+mn_0}}^J, M_J) = Tr((T_x^J)^{n+mn_0} T_{w_\Gamma}^J, M_J) = (-1)^{\ell_J(w_\Gamma)} Tr((T_x^J)^n, M_{J,\Gamma})$.

By Proposition 3.6, $Tr(T_{\tilde{w}}^n, M) = (-1)^{n\ell(\tilde{w})-n\ell(x)} Tr((T_x^{J_{\nu_x}})^n, M_{J,\Gamma})$. The “in particular” part follows from the proof of Corollary 3.7. \square

The following result is proved by Ollivier in [13, Proposition 5.2].

Lemma 4.2. *Let $J \subseteq F_0$ and $M \in R(\tilde{\mathcal{H}}_{J,0})_{\mathbf{k}}$. Then $\tilde{\mathcal{H}}_0 \otimes_{\tilde{\mathcal{H}}_{J,0}^+} M \cong \bigoplus_{d \in W_0^J} T_d \otimes M \cong \bigoplus_{d \in W_0^J} {}^t T_d \otimes M$ as vector spaces.*

Corollary 4.3. *Let $J_1 \subseteq J_2 \subseteq F_0$ and $M \in R(\tilde{\mathcal{H}}_{J_1,0})_{\mathbf{k}}$. Then*

$$\tilde{\mathcal{H}}_0 \otimes_{\tilde{\mathcal{H}}_{J_2,0}^+} (\tilde{\mathcal{H}}_{J_2,0} \otimes_{\tilde{\mathcal{H}}_{J_1,0}^+} M) \cong \tilde{\mathcal{H}}_0 \otimes_{\tilde{\mathcal{H}}_{J_1,0}^+} M.$$

4.2. Inspired by Lemma 4.1, we construct some representations of $\tilde{\mathcal{H}}_0$.

For $J \subseteq F_0$ and $\Gamma \subseteq J_{aff}$, we set $\tilde{\mathcal{H}}_{J,0}(\Gamma) = \mathcal{H}_{J,0} \rtimes \Omega_J(\Gamma)$. This is the subalgebra of $\tilde{\mathcal{H}}_{J,0}$ generated by T_s^J for $s \in J_{aff}$ and $\Omega_J(\Gamma)$.

Let $\chi \in \Omega_J(\Gamma)^\vee = \text{Hom}_{\mathbb{Z}}(\Omega_J(\Gamma), \mathbf{k}^\times)$. We extend χ as the 1-dimensional $\tilde{\mathcal{H}}_{J,0}(\Gamma)$ -module, where T_s^J acts by -1 if $s \in \Gamma$ and by 0 if $s \in J_{aff} \setminus \Gamma$. Set

$$\pi_{J,\Gamma,\chi} = \tilde{\mathcal{H}}_0 \otimes_{\tilde{\mathcal{H}}_{J,0}^+} (\tilde{\mathcal{H}}_{J,0} \otimes_{\tilde{\mathcal{H}}_{J,0}(\Gamma)} \chi).$$

4.3. Let $\aleph = \{(J, \Gamma); J \subseteq F_0, \Gamma \subseteq J_{aff}, \#W_\Gamma < +\infty\}$. We define an equivalence relation \sim and a partial order $<$ on \aleph as follows. Let $(J, \Gamma), (J', \Gamma') \in \aleph$. We say that $(J, \Gamma) \sim (J', \Gamma')$ if $J = J'$ and $\Gamma' = \tau \Gamma \tau^{-1}$ for some $\tau \in \Omega_J$. We say that $(J, \Gamma) < (J', \Gamma')$ either $J \subsetneq J'$ or $J = J'$ and $\Gamma \supsetneq \tau \Gamma \tau^{-1}$ for some $\tau \in \Omega_J$.

It is easy to see that for $(J, \Gamma) \in \aleph$, $\chi \in \Omega_J(\Gamma)^\vee$ and $\tau \in \Omega_J$, we have $\Omega_J(\tau \Gamma \tau^{-1}) = \tau \Omega_J(\Gamma) \tau^{-1}$ and $\chi \circ \text{Ad}(\tau^{-1}) \in \Omega_J(\tau \Gamma \tau^{-1})^\vee$. Moreover, $\pi_{J,\Gamma,\chi}$ and $\pi_{J,\Gamma',\chi \circ \text{Ad}(\tau^{-1})}$ are isomorphic as $\tilde{\mathcal{H}}_0$ -modules.

The main result of this section is

Theorem 4.4. *Let $(J, \Gamma) \in \aleph$ and $\chi \in \Omega_J(\Gamma)^\vee$. Let $\tilde{w} \in \tilde{W}_{\min}$ with associated standard pair (x, Γ') . Then*

- (1) *If $J \not\subseteq J_{\nu_x}$, then $Tr(T_{\tilde{w}}, \pi_{J,\Gamma,\chi}) = 0$.*
- (2) *If $J = J_{\nu_x}$ and $x \notin \Omega_J(\Gamma)$, then $Tr(T_{\tilde{w}}, \pi_{J,\Gamma,\chi}) = 0$.*
- (3) *If $J = J_{\nu_x}$ and $x \in \Omega_J(\Gamma)$, then*

$$Tr(T_{\tilde{w}}, \pi_{J,\Gamma,\chi}) = (-1)^{\ell(\tilde{w})-\ell(x)} \chi(x) \# \{\tau \in \Omega_J / \Omega_J(\Gamma); \tau^{-1} \Gamma' \tau \subseteq \Gamma\}.$$

Proof. Set $J' = J_{\nu_x}$ and $M = \tilde{\mathcal{H}}_{J,0} \otimes_{\tilde{\mathcal{H}}_{J,0}(\Gamma)} \chi$. Then $\pi_{J,\Gamma,\chi} \cong \bigoplus_{d \in W_0^{J'}} T_d \otimes M$ as vector spaces and $M = \bigoplus_{\tau \in \Omega_J / \Omega_J(\Gamma)} \mathbf{k} T_\tau^J \otimes v$ for any nonzero vector v in the 1-dimensional representation χ of $\tilde{\mathcal{H}}_{J,0}(\Gamma)$. By Proposition 3.6, to compute the character of $T_{\tilde{w}}$, it suffices to compute the character of $T_{w_{\Gamma}x^n}$ for $n \gg 0$.

Let $n_1 \in \mathbb{N}$ such that $n_1 \nu_x \in \mathbb{Z} R_J$. Set $\lambda = n_1 \nu_x$.

(1) We first consider the case where $J \not\subseteq J'$. By Proposition 3.4 (2), for $n \gg 0$, $T_{w_{\Gamma'}x^{n+n_1}} = T_{w_{\Gamma}x^n} T_{t^\lambda}$. By definition, for $s \in S_0$,

$$T_{t^\lambda} {}^\iota T_s = \begin{cases} {}^\iota T_s T_{t^\lambda}, & \text{if } s \in J'; \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $d \in W_0^J$,

$$(a) \quad T_{t^\lambda} {}^\iota T_d = \begin{cases} {}^\iota T_d T_{t^\lambda}, & \text{if } d \in W_{J'}; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, in M we have $T_{t^\lambda}(T_\tau^J \otimes v) = T_{t^\lambda}^J(T_\tau^J \otimes v) = T_\tau^J \otimes (T_{\tau^{-1}t^\lambda\tau}^J v)$.

By definition, for $\tilde{u} \in (W_J)_{aff} \rtimes \Omega_J$, $T_{\tilde{u}}^J v \neq 0$ if and only if $\tilde{u} \in W_\Gamma \rtimes \Omega_J(\Gamma)$. In particular, $\langle \nu_{\tilde{u}}, \alpha^\vee \rangle = 0$ for any $\alpha \in R_J$. Note that $\nu_{\tau^{-1}t^\lambda\tau} = w(\nu_{t^\lambda}) = n_1 w(\nu_x)$ for some $w \in W_J$. Since $J \not\subseteq J'$, there exists $\beta \in R_J$ such that $\langle \nu_x, \beta^\vee \rangle \neq 0$. Therefore $\langle \nu_{\tau^{-1}t^\lambda\tau}, w(\beta^\vee) \rangle \neq 0$ and $T_{\tau^{-1}t^\lambda\tau}^J v = 0$. Hence $Tr(T_{w_{\Gamma}x^{n+n_1}}, \pi_{J,\Gamma,\chi}) = 0$ for $n \gg 0$. By Proposition 3.6, $Tr(T_{\tilde{w}}^{n+n_1}, \pi_{J,\Gamma,\chi}) = 0$ for $n \gg 0$. By Corollary 3.7, $Tr(T_{\tilde{w}}, \pi_{J,\Gamma,\chi}) = 0$.

(2) Now we consider the case where $J = J'$. By (a), for $n \gg 0$, $T_{w_{\Gamma'}x^n} \pi_{J,\Gamma,\chi} \subseteq M$. Applying Lemma 4.1, we have $Tr(T_{\tilde{w}}, \pi_{J,\Gamma,\chi}) = (-1)^{\ell(\tilde{w}) - \ell(x)} Tr(T_x^J, T_{w_{\Gamma'}}^J M)$.

We fix a representative for each coset $\Omega_J / \Omega_J(\Gamma)$. Then $\{T_\tau^J \otimes v; \tau \in \Omega_J / \Omega_J(\Gamma)\}$ is a basis of M . For $x \in \Omega_J$, the action of T_x^J on M permutes the lines $\mathbf{k}(T_\tau^J \otimes v)$ with $\tau \in \Omega_J / \Omega_J(\Gamma)$. Moreover, the action of $T_{w_{\Gamma'}}^J$ stabilizes each line $\mathbf{k}(T_\tau^J \otimes v)$. If $x \notin \Omega_J(\Gamma)$, then there is no line $\mathbf{k}(T_\tau^J \otimes v)$ stabilized by T_x^J since Ω_J is abelian. Hence $Tr(T_{\tilde{w}}, \pi_{J,\Gamma,\chi}) = 0$ in this case.

If $x \in \Omega_J(\Gamma)$, then for any $\tau \in \Omega_J / \Omega_J(\Gamma)$, $T_x^J(T_\tau^J \otimes v) = \chi(x) T_\tau^J \otimes v$ and

$$T_{w_{\Gamma'}}^J(T_\tau^J \otimes v) = T_\tau^J \otimes T_{w_{\tau^{-1}\Gamma'\tau}}^J v = \begin{cases} (-1)^{\ell_J(w_{\Gamma'})} T_\tau^J \otimes v, & \text{if } \tau^{-1}\Gamma'\tau \subseteq \Gamma; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\dim(T_{w_{\Gamma'}}^J M) = \#\{\tau \in \Omega_J / \Omega_J(\Gamma); \tau^{-1}\Gamma'\tau \subseteq \Gamma\}$ and

$$\begin{aligned} Tr(T_{\tilde{w}}, \pi_{J,\Gamma,\chi}) &= (-1)^{\ell(\tilde{w}) - \ell(x)} \chi(x) \dim(T_{w_{\Gamma'}}^J M) \\ &= (-1)^{\ell(\tilde{w}) - \ell(x)} \chi(x) \#\{\tau \in \Omega_J / \Omega_J(\Gamma); \tau^{-1}\Gamma'\tau \subseteq \Gamma\}. \end{aligned}$$

The proof is finished. \square

Corollary 4.5. *Let $(J, \Gamma) \in \aleph$ and $\chi \in \Omega_J(\Gamma)^\vee$. Let $\tilde{w} \in \tilde{W}_{\min}$ with a standard pair (x', Γ') . If $Tr(T_{\tilde{w}}, \pi_{J, \Gamma, \chi}) \neq 0$, then $(J, \Gamma) \leqslant (J_{\nu_{x'}}, \Gamma')$.*

5. REPRESENTATIONS OF $\tilde{\mathcal{H}}_0$

Now we prove Theorem 0.2.

5.1. We first show that $\{\pi_{J, \Gamma, \chi}; (J, \Gamma) \in \aleph/\sim, \chi \in \Omega_J(\Gamma)^\vee\}$ is linearly independent in $R(\tilde{\mathcal{H}}_0)_k$.

Suppose that $\sum_{(J, \Gamma, \chi)} a_{J, \Gamma, \chi} \pi_{J, \Gamma, \chi} = 0$ for some $a_{J, \Gamma, \chi} \in \mathbb{Z}$.

Let $(J_1, \Gamma_1) \in \aleph/\sim$ be a minimal element such that $a_{J_1, \Gamma_1, \chi} \neq 0$ for some $\chi \in \Omega_{J_1}(\Gamma_1)^\vee$. Set $\Omega_{J_1}(\Gamma_1)_+ = \{x \in \Omega_{J_1}; \nu_x \in X_{\mathbb{Q}}^+\}$. It is easy to see that $\Omega_{J_1}(\Gamma_1)_+$ generates $\Omega_{J_1}(\Gamma_1)$.

By Theorem 4.4, for any $n \gg 0$ and $x \in \Omega_{J_1}(\Gamma_1)_+$,

$$\sum_{(J, \Gamma, \chi)} a_{J, \Gamma, \chi} Tr(T_{w_{\Gamma_1} x^n}, \pi_{J, \Gamma, \chi}) = \sum_{\chi \in \Omega_{J_1}(\Gamma_1)^\vee} a_{J_1, \Gamma_1, \chi} (-1)^{n\ell(\tilde{w}) - n\ell(x)} \chi^n(x) = 0.$$

Therefore $\sum_{\chi \in \Omega_{J_1}(\Gamma_1)^\vee} a_{J_1, \Gamma_1, \chi} \chi(x) = 0$. By Dedekind's lemma, $a_{J_1, \Gamma_1, \chi} = 0$ for all $\chi \in \Omega_{J_1}(\Gamma_1)^\vee$. That is a contradiction. Hence $a_{J, \Gamma, \chi} = 0$ for all (J, Γ, χ) .

5.2. Next we show that $\{\pi_{J, \Gamma, \chi}; (J, \Gamma) \in \aleph/\sim, \chi \in \Omega_J(\Gamma)^\vee\}$ spans $R(\tilde{\mathcal{H}}_0)_k$.

For any $M \in R(\tilde{\mathcal{H}}_0)_k$, let $\aleph(M)$ be the set of pairs (J_{ν_x}, Γ) in \aleph/\sim such that $Tr(T_{\tilde{w}}, M) \neq 0$ for some $\tilde{w} \in \tilde{W}_{\min}$ with an associated standard pair (x, Γ) .

We argue by induction on minimal elements in $\aleph(M)$.

If $\aleph(M) = \emptyset$, then $Tr(T_{\tilde{w}}, M) = 0$ for all $\tilde{w} \in \tilde{W}_{\min}$. By Theorem 0.1, $Tr(h, M) = 0$ for all $h \in \tilde{\mathcal{H}}_0$. Hence $M = 0$.

Now suppose that $\aleph(M) \neq \emptyset$. Let (J, Γ) be a minimal element in $\aleph(M)$. We regard $M_{J, \Gamma}$ as a virtual $\Omega_J(\Gamma)$ -module. Therefore $M_{J, \Gamma} = \sum_{\chi \in \Omega_J(\Gamma)^\vee} a_\chi \chi$ for some $a_\chi \in \mathbb{Z}$. We write $U_{J, \Gamma}$ for the $\tilde{\mathcal{H}}_0$ -module $\sum_{\chi \in \Omega_J(\Gamma)^\vee} a_\chi \pi_{J, \Gamma, \chi}$. By Lemma 4.1 and Theorem 4.4, for any $\tilde{w} \in \tilde{W}_{\min}$ with an associated standard pair $(x, \Gamma) \in \aleph/\sim$ such that $J_{\nu_x} = J$, we have

$$(a) \quad Tr(T_{\tilde{w}}, M) = (-1)^{\ell(\tilde{w}) - \ell(x)} Tr(x, M_{J, \Gamma}) = Tr(T_{\tilde{w}}, U_{J, \Gamma}).$$

Let $(J_1, \Gamma_1), \dots, (J_r, \Gamma_r)$ be the set of all minimal elements in $\aleph(M)$. Set

$$M' = M - \sum_{i=1}^r U_{J_i, \Gamma_i}.$$

By (a) and Corollary 4.5, if $\tilde{w}' \in \tilde{W}_{\min}$, with an associated standard pair (x', Γ') , satisfies $Tr(T_{\tilde{w}'}, M') \neq 0$, then $(J_{\nu_{x'}}, \Gamma') > (J_i, \Gamma_i)$ for some i . By inductive hypothesis, M' is a linear combination of

$\{\pi_{J,\Gamma,\chi}; (J,\Gamma) \in \aleph/\sim, \chi \in \Omega_J(\Gamma)^\vee\}$. So M is a linear combination of $\{\pi_{J,\Gamma,\chi}; (J,\Gamma) \in \aleph/\sim, \chi \in \Omega_J(\Gamma)^\vee\}$.

5.3. Motivated by [3], we introduce rigid modules of $\tilde{\mathcal{H}}_0$. Recall that T_Σ for $\Sigma \in \text{Cyc}(\tilde{W}_{\min})$, form a basis of $\tilde{\mathcal{H}}_0$. Set

$$\begin{aligned}\overline{\mathcal{H}}_0^{\text{rig}} &= \bigoplus_{\Sigma \in \text{Cyc}(\tilde{W}_{\min}), J_{\nu_\Sigma} = F_0} \mathbb{Z}T_\Sigma, \\ \overline{\tilde{\mathcal{H}}}_0^{\text{nrig}} &= \bigoplus_{\Sigma \in \text{Cyc}(\tilde{W}_{\min}), J_{\nu_\Sigma} \neq F_0} \mathbb{Z}T_\Sigma.\end{aligned}$$

We call $\overline{\mathcal{H}}_0^{\text{rig}}$ the rigid part of the cocenter and $\overline{\tilde{\mathcal{H}}}_0^{\text{nrig}}$ the non-rigid part of the cocenter.

Let $M \in R(\tilde{\mathcal{H}}_0)_k$. We say M is rigid if $\text{Tr}(\overline{\tilde{\mathcal{H}}}_0^{\text{nrig}}, M) = 0$.

Proposition 5.1. *Let $M \in R(\tilde{\mathcal{H}}_0)_k$. Then M is rigid if and only if $M \in \bigoplus_{(F_0, \Gamma) \in \aleph/\sim, \chi \in \Omega(\Gamma)^\vee} \mathbb{Z}\pi_{F_0, \Gamma, \chi}$.*

Remark. By Clifford's theory, the $\tilde{\mathcal{H}}_0$ -modules $\pi_{F_0, \Gamma, \chi}$ for $(F_0, \Gamma) \in \aleph/\sim$ and $\chi \in \Omega(\Gamma)^\vee$ are distinct simple modules.

Proof. By Theorem 4.4, $\pi_{F_0, \Gamma, \chi}$ is rigid. On the other hand, assume $M = \sum_{(J, \Gamma) \in \aleph/\sim, \chi \in \Omega_J(\Gamma)^\vee} a_{J, \Gamma, \chi} \pi_{J, \Gamma, \chi}$ with each $a_{J, \Gamma, \chi} \in \mathbb{Z}$. Let (J', Γ') be a minimal element such that $a_{J', \Gamma', \chi'} \neq 0$ for some $\chi' \in \Omega_{J'}(\Gamma')^\vee$. By the same argument as in §5.2, we see that $\text{Tr}(T_{w_{\Gamma'} t^\mu}, M) \neq 0$ for some $\mu \in X^+(J')$. Hence M is nonrigid unless $J' = F_0$, that is, $M \in \bigoplus_{(F_0, \Gamma) \in \aleph/\sim, \chi \in \Omega(\Gamma)^\vee} \mathbb{Z}\pi_{F_0, \Gamma, \chi}$. \square

5.4. Let $\tilde{w} = wt^\lambda \in \tilde{W}$ with $\lambda \in X$ and $w \in W_0$. Write $\lambda = \mu_1 - \mu_2$ with $\mu_1, \mu_2 \in X^+$. Following Vignéras, we define

$$E_{\tilde{w}} = q^{\frac{1}{2}(\ell(\mu_2) - \ell(\mu_1) - \ell(w) + \ell(\tilde{w}))} T_{wt^{\mu_1}} T_{t^{\mu_2}}^{-1} \in \tilde{\mathcal{H}}_q,$$

which dose not depend on the choices of μ_1 and μ_2 . We still denote by $E_{\tilde{w}}$ its image in $\tilde{\mathcal{H}}_0$. By [17], the set $\{E_{\tilde{w}}; \tilde{w} \in \tilde{W}\}$ forms a basis of $\tilde{\mathcal{H}}_0$.

Lemma 5.2. *Let $x, y \in \tilde{W}$ with $\ell(x) \leq \ell(y)$. Then*

$$\begin{aligned}q^{\frac{1}{2}(\ell(x) - \ell(y) + \ell(yx))} T_y T_{x^{-1}}^{-1} &\in \left(\bigoplus_{z \in \tilde{W}, \ell(z) \geq \frac{1}{2}(\ell(y) - \ell(x) + \ell(yx))} \mathbb{Z}T_z \right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}_q, \\ q^{\frac{1}{2}(\ell(y) - \ell(x) + \ell(xy))} T_x T_{y^{-1}}^{-1} &\in \left(\bigoplus_{z \in \tilde{W}, \ell(z) \geq \frac{1}{2}(\ell(y) - \ell(x) + \ell(xy))} \mathbb{Z}^{\ell}T_z \right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}_q.\end{aligned}$$

Proof. We prove the first statement. The second one can be proved in the same way.

We argue by induction on $\ell(x)$. If $\ell(x) = 0$, then statement is obvious. Assume $\ell(x) \geq 1$ and the statement holds for any x' with $\ell(x') < \ell(x)$. Let $s \in S_{\text{aff}}$ such that $sx < x$.

If $ys < y$, then

$$q^{\frac{1}{2}(\ell(x) - \ell(y) + \ell(yx))} T_y T_{x^{-1}}^{-1} = q^{\frac{1}{2}(\ell(sx) - \ell(ys) + \ell(yssx))} T_{ys} T_{(sx)^{-1}}^{-1}$$

and $\ell(y) - \ell(x) + \ell(yx) = \ell(ys) - \ell(sx) + \ell(yssx)$. The statement follows from induction hypothesis.

If $ys > y$, then

$$\begin{aligned} q^{\frac{1}{2}(\ell(x)-\ell(y)+\ell(yx))}T_y T_{x^{-1}}^{-1} &= q^{\frac{1}{2}(\ell(sx)-\ell(ys)+\ell(yx))}T_{ys} T_{(sx)^{-1}}^{-1} \\ &\quad + q^{\frac{1}{2}(\ell(sx)-\ell(y)+\ell(yx)-1)}(1-q)T_y T_{(sx)^{-1}}^{-1}. \end{aligned}$$

By inductive hypothesis,

$$q^{\frac{1}{2}(\ell(sx)-\ell(ys)+\ell(yx))}T_{ys} T_{(sx)^{-1}}^{-1} \in (\bigoplus_{z \in \tilde{W}, \ell(z) \geq \frac{1}{2}(\ell(y)-\ell(x)+\ell(yx))} \mathbb{Z}T_z) + q\mathbb{Z}[q]\tilde{\mathcal{H}}_q.$$

Let α be the simple root associated to s and $\beta = x^{-1}(\alpha)$. Then $\beta < 0$ since $sx < x$ and $yx(\beta) = y(\alpha) > 0$ since $ys > y$. Hence $ysx = yxs_\beta < yx$. Therefore, $\ell(ysx) \leq \ell(yx) - 1$ and $\ell(sx) - \ell(y) + \ell(yx) - 1 \geq \ell(sx) - \ell(y) + \ell(ysx)$.

If $\ell(ysx) < \ell(yx) - 1$, then $\ell(sx) - \ell(y) + \ell(yx) - 1 > \ell(sx) - \ell(y) + \ell(ysx)$ and by inductive hypothesis, $q^{\frac{1}{2}(\ell(sx)-\ell(y)+\ell(yx)-1)}(1-q)T_y T_{(sx)^{-1}}^{-1} \in q\mathbb{Z}[q]\tilde{\mathcal{H}}_q$ and the statement holds in this case.

If $\ell(ysx) = \ell(yx) - 1$, then $\ell(y) - \ell(x) + \ell(yx) = \ell(y) - \ell(sx) + \ell(ysx)$ and by inductive hypothesis,

$$\begin{aligned} q^{\frac{1}{2}(\ell(sx)-\ell(y)+\ell(yx)-1)}(1-q)T_y T_{(sx)^{-1}}^{-1} &\in (\bigoplus_{z \in \tilde{W}, \ell(z) \geq \frac{1}{2}(\ell(y)-\ell(sx)+\ell(ysx))} \mathbb{Z}T_z) + q\mathbb{Z}[q]\tilde{\mathcal{H}}_q \\ &= (\bigoplus_{z \in \tilde{W}, \ell(z) \geq \frac{1}{2}(\ell(y)-\ell(x)+\ell(yx))} \mathbb{Z}T_z) + q\mathbb{Z}[q]\tilde{\mathcal{H}}_q \end{aligned}$$

The statement also holds in this case. \square

Corollary 5.3. *Let $\Gamma \subseteq S_{aff}$ with $\#W_\Gamma < \infty$ and $\tilde{w} \in \tilde{W}$ with $\ell(\tilde{w}) > 2\#W_\Gamma$. Then in $\tilde{\mathcal{H}}_0$,*

$$E_{\tilde{w}} \in \bigoplus_{z \in \tilde{W}, \text{supp}(z) \not\subseteq \Gamma} \mathbb{Z}T_z \text{ or } E_{\tilde{w}} \in \bigoplus_{z \in \tilde{W}, \text{supp}(z) \not\subseteq \Gamma} \mathbb{Z}^\nu T_z.$$

Proof. By definition, $E_{\tilde{w}} = q^{\frac{1}{2}(\ell(x)-\ell(y)+\ell(yx))}T_y T_{x^{-1}}^{-1}$ for some $x, y \in \tilde{W}$ such that $yx = \tilde{w}$. Applying Lemma 5.2, we see that $E_{\tilde{w}} \in \bigoplus_{z \in \tilde{W}, \ell(z) > \#W_\Gamma} \mathbb{Z}T_z$ if $\ell(x) \leq \ell(y)$ and $E_{\tilde{w}} \in \bigoplus_{z \in \tilde{W}, \ell(z) > \#W_\Gamma} \mathbb{Z}^\nu T_z$ if $\ell(y) \leq \ell(x)$. The statement follows by noticing that $\text{supp}(z) \not\subseteq \Gamma$ if $\ell(z) > \#W_\Gamma$. \square

Proposition 5.4. *Let $M \in R(\tilde{\mathcal{H}}_0)$. The following conditions are equivalent:*

- (1) $E_{\tilde{w}}M = 0$ for $\tilde{w} \in \tilde{W}$ with $\ell(\tilde{w}) \gg 0$.
- (2) $\text{Tr}(\overline{\tilde{\mathcal{H}}_0}^{nss}, M) = 0$, where $\overline{\tilde{\mathcal{H}}_0}^{nss} = \overline{\tilde{\mathcal{H}}_0}^{nrig} + \iota(\overline{\tilde{\mathcal{H}}_0}^{nrig})$.
- (3) $M \in \bigoplus_{(F_0, \Gamma), (F_0, F_0 \setminus \Gamma) \in \aleph, \chi \in \Omega_{F_0}(\Gamma)} \mathbb{Z}\pi_{F_0, \Gamma, \chi}$.

Remark. Condition (1) is the one of the equivalent definitions of supersingular modules due to Ollivier [14, Proposition 5.4] and Vignéras [20, Definition 6.10]. The equivalence between (1) and (3) was also proved in [14, Theorem 5.14] and in [20, Theorem 6.18].

Proof. (1) \Rightarrow (2). Let $\tilde{w} \in \tilde{W}_{\min}$ such that $J_{\nu_{\tilde{w}}} \subsetneq F_0$. Let (x, Γ) be a standard pair associated to \tilde{w} . Choose $n_0, m_0 \in \mathbb{Z}_{>0}$ such that $n_0\nu_x \in X^+$ and $w_{\Gamma}x^{m_0} \in \tilde{W}_J^+$. Then $\ell(w_{\Gamma}x^{m_0+r+k n_0}) = \ell(w_{\Gamma}x^{m_0+r}) + \ell(x^{k n_0})$ for $k \in \mathbb{N}$ and $0 \leq r \leq n_0 - 1$. Thus $T_{w_{\Gamma}x^{m_0+r+k n_0}} = T_{w_{\Gamma}x^{m_0+r}} E_{x^{k n_0}}$. By assumption, we have $T_{w_{\Gamma}x^n} M = 0$ for $n \gg 0$. Applying Proposition 3.6, $Tr(T_{\tilde{w}}^n, M) = \pm Tr(T_{w_{\Gamma}x^n}, M) = 0$ and hence $Tr(T_{\tilde{w}}, M) = 0$. The equality $Tr({}^t T_{\tilde{w}^{-1}}, M) = 0$ follows in a similar way by noticing that ${}^t T_{x^{-k n_0}} = E_{x^{-k n_0}}$ for $k \in \mathbb{N}$.

(2) \Rightarrow (3). By Proposition 5.1, M and its pullback ${}^t M$ via ι lie in the \mathbb{Z} -span of $\{\pi_{F_0, \Gamma, \chi}; (F_0, \Gamma) \in \aleph, \chi \in \Omega(\Gamma)^{\vee}\}$. By definition ${}^t \pi_{F_0, \Gamma, \chi} = \pi_{F_0, S_{\text{aff}} \setminus \Gamma, \chi}$. Thus M also lies in the \mathbb{Z} -span of $\{\pi_{F_0, \Gamma, \chi}; (F_0, \Gamma \subseteq S_{\text{aff}}) \in \aleph, \chi \in \Omega(\Gamma)^{\vee}\}$. Therefore, M lies in the \mathbb{Z} -span of $\{\pi_{F_0, \Gamma, \chi}; (F_0, \Gamma), (F_0, F_0 \setminus \Gamma) \in \aleph, \chi \in \Omega(\Gamma)^{\vee}\}$.

(3) \Rightarrow (1). Let $\Gamma \subseteq S_{\text{aff}}$. By definition, $T_x \pi_{F_0, \Gamma, \chi} = {}^t T_x \pi_{F_0, \Gamma, \chi} = 0$ for any $x \in \tilde{W}$ such that $\text{supp}(x) \not\subseteq \Gamma$ and $\text{supp}(x) \not\subseteq S_{\text{aff}} \setminus \Gamma$. Assume $\#W_{\Gamma}, \#W_{S_{\text{aff}} \setminus \Gamma} < +\infty$. Applying Corollary 5.3, $E_{\tilde{w}} \pi_{F_0, \Gamma, \chi} = 0$ for $\tilde{w} \in \tilde{W}$ with $\ell(\tilde{w}) > 2\#W_{\Gamma}, 2\#W_{S_{\text{aff}} \setminus \Gamma}$. \square

ACKNOWLEDGEMENT

The first-named author was introduced to the representations of affine 0-Hecke algebras by Marie-France Vignéras, who explained the beauty and importance of supersingular modules and encouraged the author to apply the method in [3] to the study of affine 0-Hecke algebras. It is a great pleasure to thank her. The authors also would like to thank Dan Ciubotaru and George Lusztig for many useful discussions on affine Hecke algebras, and to thank Noriyuki Abe for sending us some lecture notes on modular Iwahori-Hecke algebras.

REFERENCES

- [1] N. Abe, *Mod p parabolic induction for Pro-p-Iwahori Hecke algebra*, arXiv:1406.1003.
- [2] C. J. Bushnell and P. C. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via type*, Proc. London Math. Soc. 77 (3) (1998), 582–634.
- [3] D. Ciubotaru and X. He, *Cocenters and representations of affine Hecke algebra*, arXiv:1409.0902.
- [4] X. He, *Minimal length elements in some double cosets of Coxeter groups*, Adv. Math. 215 (2007), no. 2, 469–503.
- [5] _____, *Geometric and homological properties of affine Deligne-Lusztig varieties*, Ann. Math. 179 (2014), 367–404.
- [6] _____, *Minimal length elements of extended affine Weyl group, I*, arXiv:1004.4040, preprint.
- [7] _____, *Centers and cocenters of 0-Hecke algebras*, preprint.
- [8] X. He and S. Nie, *Minimal length elements of finite Coxeter group*, Duke Math. J. 161 (2012), 2945–2967.
- [9] _____, *Minimal length elements of extended affine Weyl group*, Compos. Math. 150 (2014), 1903–1927.

- [10] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 5–48.
- [11] D. Kazhdan, G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), no. 1, 153–215.
- [12] P. N. Norton, *0 -Hecke algebras*, J. Austral. Math. Soc. Ser. A **27** (1979), no. 3, 337–357.
- [13] R. Ollivier, *Parabolic Induction and Hecke modules in characteristic p for p -adic GL_n* , Algebra and Number Theory **4** (2010), 701–742.
- [14] ———, *Compatibility between Satake and Bernstein-type isomorphisms in characteristic p* , Algebra and Number Theory **8** (2014), 1071–1111.
- [15] M. Reeder, *Isogenies of Hecke algebras and a Langlands correspondence for ramified principal series representations*, Represent. Theory **6** (2002), 101–126.
- [16] M.-F. Vignéras, *Induced R -representations of p -adic reductive groups*, Selecta Math. (N.S.) **4** (4) (1998), 549–623.
- [17] ———, *Pro- p -Iwahori Hecke ring and supersingular $\bar{\mathbb{F}}_p$ -representations*, Math. Ann. **331** (3) (2005), 523–556. Erratum in **333** (2), 699–701.
- [18] ———, *The pro- p -Iwahori-Hecke algebra of a reductive p -adic group I*, preprint.
- [19] ———, *The pro- p -Iwahori-Hecke algebra of a reductive p -adic group II*, preprint.
- [20] ———, *The pro- p -Iwahori-Hecke algebra of a reductive p -adic group III*, preprint.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK,
MD 20742, USA AND DEPARTMENT OF MATHEMATICS, HKUST, HONG KONG
E-mail address: xuhuahe@math.umd.edu

INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS
SCIENCE, CHINESE ACADEMY OF SCIENCES, 100190, BEIJING, CHINA
E-mail address: niesian@amss.ac.cn